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Bilinear operators and the power series for the Weierstrass σ function

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Abstract. We use the bilinear operator formalism to derive a new representation of the power series for the Weierstrass σ function.

It is not generally known that Baker solved a number of nonlinear integrable partial differential equations in 1907 [2]. In the course of an investigation of ultra-elliptic functions, he wrote certain relations between them, in which with hindsight it is possible to recognize some integrable hierarchies of soliton theory. Among other things, he introduced the bilinear operator method, a technique more recently discovered independently (and extensively developed) by Hirota (cf [5]). Baker was concerned with the σ function associated with a genus-2 hyperelliptic algebraic curve, in today's language such a function is known as the τ function of the curve. It is the natural generalization of the classical genus-1 σ function introduced by Weierstrass in his study of elliptic functions. Our work is a part of a general programme (Buchstaber *et al* [3, 4]) to extend Baker's work to algebraic curves of higher genera, but it is amusing to note that Baker's techniques give an elegant formula for the power-series expansion of the elliptic σ function. The derivation seems to be technically simpler than that given by Weierstrass [7] and appears to be new. This small but instructive application forms the basis of this short paper.

First we present a few words to enlarge on our historical remarks. On p 88 of [2] we find a set of partial differential equations, the first of which is

$$\wp_{2222} = 6\wp_{22}^2 + \tfrac{1}{2}\lambda_3 + \lambda_4\wp_{22} + 4\wp_{21}. \qquad (1)$$

The $\wp$'s are functions of two variables and the subscripts 1, 2 denote partial differentiation with respect to variables 1 and 2, respectively. Setting variable $1 = t'$, $2 = x$ and differentiating with respect to x gives

$$\wp_{xxxxx} = 12\wp_{xx}\wp_{xxx} + \lambda_4\wp_{xxx} + 4\wp_{t'xx}.$$

Now take $\lambda_4 = 0$, $t' = -4t$, $u(x,t) = \wp_{xx}(x,t)$, to obtain

$$u_t + 12uu_x + u_{xxx} = 0$$

the well known KdV equation. The connection between Baker's work, the KdV equation, Hirota's bilinear form and vertex operator techniques was first pointed out in [3].

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In Baker's approach, $\wp_{i,j}$ and $\wp_{i,j,k,\ell}$ are defined as

$$\wp_{i,j} = -\frac{\partial^2}{\partial z_i \partial z_j} \ln \sigma(z_1, z_2) \qquad \wp_{i,j,k,\ell} = -\frac{\partial^4}{\partial z_i \partial z_j \partial z_k \partial z_\ell} \ln \sigma(z_1, z_2)$$

where $\sigma(z_1, z_2)$ is a genus-2 σ function associated with the hyperelliptic curve $y^2 = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \lambda_4 x^4 + 4x^5$. In order to develop a power series for this function, Baker shows that it is a solution of the following equation:

$$\tfrac{1}{3}\Delta_h^4 \sigma\sigma' = \text{lower-order terms}$$

where $\Delta_h = h_1(\partial/\partial z_1 - \partial/\partial z_1') + h_2(\partial/\partial z_2 - \partial/\partial z_2')$ and $\sigma'(z_1, z_2) = \sigma(z_1', z_2')$, with the usual convention that z_i' is replaced by z_i after the derivatives have been carried out. In this formulation, equation (1) is recovered as the term with coefficient h_2^4. With some further powerful algebraic techniques, Baker derives a power series for $\sigma(z_1, z_2)$ which is convergent for all finite $z_1, z_2 \in \mathcal{C}^2$.

We now apply these techniques to the genus-1 case. The Weierstrass σ function is connected to the Weierstrass elliptic function $\wp$ by

$$\wp(x) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln \sigma(x) \tag{2}$$

where $\wp(x)$ satisfies the well known relations

$$\left(\frac{\mathrm{d}\wp(x)}{\mathrm{d}x}\right)^2 = 4\wp(x)^3 - g_2\wp(x) - g_3 \tag{3}$$

$$\frac{\mathrm{d}^2\wp(x)}{\mathrm{d}x^2} = 6\wp(x)^2 - \tfrac{1}{2}g_2. \tag{4}$$

Clearly (4) is just the derivative of (3) and contains no new information, however, we will find it useful to use both equations. We note also that Mitra [6] found it convenient to use both (3) and (4) to derive coefficients of the power-series expansion of the Weierstrass $\wp$-function, although his method is otherwise unrelated to our own.

In his original 1882 paper [7], Weierstrass carried out some remarkable manipulations to derive new equations involving the derivatives of the equations with respect to the parameters g_2 and g_3. These modular equations are very important in their own right. With their help he arrived at the following double-summation formula:

$$\sigma(z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n}\left(\tfrac{1}{2}g_2\right)^m (2g_3)^n \frac{z^{4m+6n+1}}{(4m+6n+1)!}$$

where the $a_{m,n}$ satisfy the recurrence relation

$$a_{m,n} = 3(m+1)a_{m+1,n-1} + \tfrac{16}{3}(n+1)a_{m-2,n+1} - \tfrac{1}{3}(2m+3n-1)(4m+6n-1)a_{m-1,n}$$

with $a_{0,0} = 0$ and $a_{i,j} = 0$ if $i < 0$ or $j < 0$. These formulae are reproduced in Abramowitz and Stegun (sections 15.5.6–15.5.8 of [1]).

For our alternative approach we first note that if (2) is inserted into (4) we obtain the following differential equation for $\sigma(x)$:

$$-\sigma_{x,x,x,x}\sigma + 4\sigma_{x,x,x}\sigma_x - 3\sigma_{x,x}^2 + \tfrac{1}{2}g_2\sigma^2 = 0$$

where $\sigma_x = \mathrm{d}\sigma(x)/\mathrm{d}x$, etc. It is not difficult to show that this can be written in bilinear form as

$$(\Delta^4 - g_2)\sigma\sigma' = 0 \tag{5}$$

where $\Delta = \mathrm{d}/\mathrm{d}x - \mathrm{d}/\mathrm{d}x'$. Since σ is an odd function, conventionally normalized with the first term in its expansion given by x, we write

$$\sigma(x) = x + \sum_{n=1}^{\infty} c_{2n+1} x^{2n+1}.$$

Inserting this into (5) and collecting terms in x^{2n} we find

$$\Delta^4 \sum_{i=1}^{n+2} x^{2n+5-2i} (x')^{2i-1} c_{2n+5-2i} c_{2i-1} = g_2 \sum_{i=1}^{n} x^{2n+1-2i} (x')^{2i-1} c_{2n+1-2i} c_{2i-1} \tag{6}$$

where $c_1 = 1$, and we have anticipated the following result that follows after some algebraic manipulation:

$$\Delta^4 x^n (x')^m = \Delta^4 x^m (x')^n = b_{n,m} x^{(n+m-4)}$$

with

$$b_{n,\ell} = (n+\ell)^4 - 6(n+\ell)^3 + (11 - 8n\ell)(n+\ell)^2 + (24n\ell - 6)(n+\ell) + 16n\ell(n\ell - 2)$$

so (6) becomes, after putting $x' = x$,

$$\sum_{i=1}^{n+2} b_{2n+5-2i,2i-1} c_{2n+5-2i} c_{2i-1} = g_2 \sum_{i=1}^{n} c_{2n+1-2i} c_{2i-1}.$$

Because of the symmetry of $b_{n,m}$, when n is even, this leads to

$$b_{2n+3,1} c_{2n+3} = -\sum_{i=1}^{n/2} b_{2n+3-2i,2i+1} c_{2n+3-2i} c_{2i+1} + g_2 \sum_{i=1}^{n/2} c_{2n+1-2i} c_{2i-1} \tag{7}$$

and when n is odd

$$\begin{aligned} b_{2n+3,1} c_{2n+3} = &- \sum_{i=1}^{(n-1)/2} b_{2n+3-2i,2i+1} c_{2n+3-2i} c_{2i+1} - \tfrac{1}{2} b_{n+2,n+2} c_{n+2}^2 \\ &+ g_2 \sum_{i=1}^{(n-1)/2} c_{2n+1-2i} c_{2i-1} + \tfrac{1}{2} g_2 c_n^2. \end{aligned} \tag{8}$$

Now $b_{2n+3,1} = 4(n-2)(2n+3)(2n+1)(n+1)$, so for $n \geqslant 0, n \neq 2$, equations (7) and (8) can easily be solved for c_{2n+3} in terms of coefficients of lower order. In particular, we find immediately from the cases $n = 0$ and 1 that $c_3 = 0$ and $c_5 = -g_2/240$, respectively. The case $n = 2$ gives no information about c_7 due to the vanishing of $b_{2n+3,1}$, and all higher cases for $n > 3$ require this coefficient. It should not be a surprise that we cannot solve the whole series immediately since (5) does not involve g_3. (It is interesting that Baker finds a similar problem in the genus-2 case at this point.)

To solve for c_7, we need to insert the series into the σ equation corresponding to (3). Balancing the terms in x^7 we find that $c_7 = -g_3/840$. Now we can proceed with (7) and (8) for $n > 2$ to give the remaining c_{2n+3} coefficients up to any desired order. In explicit form these are finally as follows. For odd $n \geqslant 1$

$$\begin{aligned} c_{2n+3} = \frac{1}{b_{2n+3,1}} \Bigg(&\sum_{i=1}^{(n-1)/2} \left(-b_{2n+3-2i,2i+1}\, c_{2n+3-2i}\, c_{2i+1} + g_2 c_{2n+1-2i}\, c_{2i-1} \right) \\ &- 6(n+2)(n+1)\, c_{n+2}^2 + \tfrac{1}{2} g_2 c_n^2 \Bigg) \end{aligned}$$

where we have inserted the explicit equation for $b_{n+2,n+2}$. For even n, $n \neq 2$

$$c_{2n+3} = \frac{1}{b_{2n+3,1}} \sum_{i=1}^{n/2} \left(-b_{2n+3-2i,2i+1}\, c_{2n+3-2i}\, c_{2i+1} + g_2 c_{2n+1-2}\, c_{2i-1} \right)$$

with $b_{n,\ell}$ and c_1, c_7 as defined above.

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